REAL ANALYSIS HOMEWORK 4

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1. Problem 1

By definition, $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k := A$. Hence, suppose that $x \in A$. By definition, there exists n such that $x \in \bigcap_{k\geq n} A_k$. However, by definition of intersection, this means that for all $k \geq n$, $x \in A_k$. Since n is finite, the result follows immediately, since there exists n such that $x \in A_k$ for all $k \geq n$, or in other words, the set of positive integers m for which $x \notin A_m$ is finite.

2. Problem 2

To show $\lim_{n\to\infty} A_n := A$ exists, we must show that $\liminf_{n\to\infty} A_n := A^-$ and $\limsup_{n\to\infty} A_n := A^+$ exist and are equal.

Suppose now that $x \in A^-$. Then, by the result of problem 1, $x \in (-1/n, 1-1/n)$ for infinitely many n. Letting $n \to \infty$, however, this implies that $x \in (0,1)$, so $A^- \subset (0,1)$. To show the reverse inclusion, suppose that $x \notin A^-$. Then, there is some subsequence n_k such that $x \notin (1/n_k, 1-1/n_k)$ for all $k \in \mathbb{Z}$. Letting $k \to \infty$, we see that $x \notin (0,1)$, so that $A^- = (0,1)$ by contraposition.

Suppose now that $x \in A^+$. Then, as shown in class, $x \in A_n$ for infinitely many n. In other words, there exists a subsequence n_k such that $x \in (-1/n_k, 1-1/n_k)$. Letting $k \to \infty$, we see $x \in (0,1)$. Conversely,

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if $x \in (0,1)$, then there exists δ such that $B_\delta(x) \subset (0,1)$. Now, find N such that $1/N < \delta$. Then $x \in A_n$ for all $n \geq N$, so that $x \in A^+$ (since $x \in A_n$ for infinitely many n), and $A^+ = (0,1)$.

Thus, by the above we see that $\lim_{n\to\infty} A_n = (0,1)$.

3. Problem 3

Suppose that $f_n \to f$ pointwise and define the following sets:

$$F(t) := \{x \in \mathbb{R} : \ f(x) \leq t\}$$
$$G(t) := \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \ f_n(x) < t + 1/m\}$$

Then if $x \in \mathbb{R}$, given any integer m, there is k such that $|f_n(x) - f(x)| < 1/m$ for all $n \geq k$ by definition of pointwise convergence. Now, suppose that $x \in F(t)$, so that $f(x) \leq t$. Then, we see that $f_n(x) < t + 1/m$. However, this shows that for all m, there exists k such that for all $n \geq k$, $f_n(x) < t + 1/m$. But this is precisely the condition for $x \in G(t)$, so $F(t) \subset G(t)$.

Conversely, let $x \in G(t)$. Then by definition, for all m, there exists k such that for all $n \geq k$, $f_n(x) < t + 1/m$. Letting $n \to \infty$, we see that $f(x) \leq t + 1/m$, and since the left hand side of this inequality is independent of m, let $m \to \infty$ on the right to conclude $f(x) \leq t$. Thus $x \in F(t)$, so that $F(t) = G(t)$, as desired.

4. Problem 4

Let $\epsilon > 0$ and enumerate the elements of E as $\{x_1, x_2, \dots\}$. For each $x_n \in E$, associate the interval $I_n := \Big(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\Big)$. Obviously the sets I_n constitute an open cover of E, so that

$$m^*(E) \leq m^*(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

Since ϵ is arbitrary, we conclude $m^*(E) \leq 0$, so $m^*(E) = 0$.

5. Problem 5

(a). Let $\{I_n\}$ be an open cover of E. Then, the set of translates $\{I_n+h\}$ for any $h \in \mathbb{R}$ is an open cover of $E+h$, with $m^*(I_n+h) = m^*(I_n)$, since if $I_n = (a,b)$, then $I_n + h = (a+h, b+h)$ has outer measure $b+h-(a+h) = b-a = m^*(I_n)$. Using this, given $\epsilon > 0$, we can find an open cover $\{I_n\}$ such that

$$m^*(E) + \epsilon > \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} m^*(I_n+h) \geq m^*(E+h)$$

Since ϵ is arbitrary, $m^*(E) \geq m^*(E+h)$. To find the reverse inequality, merely note that $E = (E+h)-h$. Hence defining $E' := E+h$, we have that $m^*(E') \geq m^*(E'-h)$. But this just says $m^*(E+h) \geq m^*(E)$, so they are in fact equal.

For $m^*(E)$ infinite, we find by similar reasoning that $m^*(E+h) \geq m^*(E)$, and hence $m^*(E+h)$ is infinite as well.

(b). Suppose that E is measurable. Then, given another set T, note that $(T+h) \cap (E+h) = T \cap E + h$, and $(E+h)^c = E^c + h$. Using this and the translation invariance just proved in part (a):

$$\begin{aligned} m^*(T+h) &\leq m^*((T+h) \cap (E+h)) + m^*((T+h) \cap (E+h)^c) \\ &= m^*(T \cap E + h) + m^*(T \cap E^c + h) \\ &= m^*(T \cap E) + m^*(T \cap E^c) \\ &= m^*(T) = m^*(T+h) \end{aligned} \tag{5.1}$$

And we see that $(E+h)$ is measurable (indeed, this is actually equivalence by identical reasoning as in part (a)).